Filomat 29:4 (2015), 751–756
DOI 10.2298/FIL1504751R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remark on Sheffer Polynomials

S. J. Rapeli[a], Pratik Shah[b], A. K. Shukla[a]

[a]*Department of Applied Mathematics & Humanities, S. V. National Institute of Technology, Surat-395 007, India*
[b]*Department of Mathematics, C.K. Pithawalla College of Engineering & Technology,Surat-395 007, India*

Abstract. This paper deals with some theorems on Sheffer A-type zero polynomial sets.

1. Introduction

A polynomial set $p_n(x)$ is said to be of Sheffer A-type zero if and only if it has a generating function in the form [3, 12, 13] as

$$A(t)\exp(xG(t)) = \sum_{n=0}^{\infty} p_n(x)t^n,$$

where $A(t)$ and $G(t)$ are two formal power series

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad a_0 \neq 0;$$

$$G(t) = \sum_{n=0}^{\infty} g_n t^{n+1}, \qquad g_0 \neq 0;$$

and $J(D)p_0(x) = 0$ and $Jp_n(x) = p_{n-1}(x)$, $n \geq 1$; where $J(D)$ is defined as

$$J = J(D) = \sum_{k=0}^{\infty} a_k D^{k+1}, \quad a_0 \neq 0 \quad \text{and} \quad D \equiv \frac{d}{dx}.$$

Al Salam and Verma [1] gave the generalized Sheffer polynomials by considering $\phi_n(x)$ as a Sheffer A-type zero

$$\sum_{i=1}^{r} A_i(t)\exp((xG(\varepsilon_i(t)) = \sum_{n=0}^{\infty} \phi_n(x)t^n,$$

2010 *Mathematics Subject Classification*. 33C65, 33F99, 44A45
Keywords. Appell sets, Differential operator, Sheffer polynomials
Received: 11 October 2013; Revised: 21 January 2014; Accepted: 05 February 2014
Communicated by Hari M. Srivastava
First Author is thankful to Sardar Vallabhbhai National Institute of Technology, Surat, India for awarding Senior Research Fellowship.
Email addresses: shrinu0711@gmail.com (S. J. Rapeli), pratikshah8284@yahoo.co.in (Pratik Shah), aks@ashd.svnit.ac.in (A. K. Shukla)

where

$$J(D) = \sum_{k=0}^{\infty} c_k D^{k+r}, \qquad J(D)\phi_n(x) = \phi_{n-r}(x), \qquad (n = r, r+1, ...)$$

Thorne [21] obtained an interesting characterization of Appell polynomials by means of Stieltjes integral. Appell sets [17] are hold following equivalent condition:
(i)$p'_n(x) = p_{n-1}(x), n = 0, 1, 2, ...$
(ii)There exists a formal power series $A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad (a_0 \neq 0)$ such that

$$A(t) \exp(xt) = \sum_{n=0}^{\infty} p_n(x)t^n.$$

Osegove [14] gave the generalization of Appell sets in a different direction. He studied polynomial sets and hold the following property

$$D^r p_n(x) = p_{n-r}(x), \qquad n \geq r,$$

where r is a (fixed) positive integer.

Huff and Rainville [11] proved the necessary and sufficient condition for polynomial $p_n(x)$. If polynomial $p_n(x)$ is generated by $A(t)\psi(xt)$ then a necessary and sufficient condition for $p_n(x)$, be a Sheffer A-type $m, m > 0$, if $\psi(xt) = {}_0F_m[-; b_1, b_2, \cdots, b_m; \alpha xt]$, where α is a nonzero constant.

Goldberg [10] generalized the above result and proved, if the polynomial set $p_n(x)$ is generated by $A(t)\psi(xB(t))$ then a necessary and sufficient condition for $p_n(x)$ to be a Sheffer A-type $m, m > 0$, is that there exist a positive number r which divides m and numbers $b_1, b_2, \cdots, b_r$ (none zero nor negative integers) such that $p_n(x)$ is σ -type zero for $\sigma = D\prod_{k=1}^{r}(xD + b_k - 1), \;\; D \equiv \frac{d}{dx}$.

Bretti et al.[5] gave Laguerre type Exponentials and generalized Appell polynomials and Dattoli [8] studied the Appell complementary forms. Khan and Raza [20] discussed the families of Legendre-Sheffer polynomials corresponding to two different forms of 2-variable Legendre polynomials. Youn and Yang [22] obtained a differential equation and recursive formulas of Sheffer polynomial sequences utilizing matrix algebra. Dattoli et al.[7] studied Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials. Bor et al.[4] studied on new application of certain generalized power increasing sequences and some interesting results on Laguerre type polynomials were discussed by Djordjević [9].
Let $p_n^{(\alpha)}(x)$ be a simple polynomial set and has following generating function [6,19]

$$(1-t)^{-\alpha}F(x,t) = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x)t^n, \tag{1}$$

where $F(x, t)$ is independent on parameter α .
If $F(x, t) = (1 - t)^{-1} \exp(\frac{-xt}{1-t})$ then this gives the generalized Laguerre polynomials $p_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)$. [16]

2. Main Results

First we prove the following Lemmas.

Lemma 1: The polynomial set $p_n^{(\alpha-\beta n)}(x)$ is generated by

$$\frac{(1+u(t))^{\alpha}}{1+\beta u(t)}F(x, u(t)[1+u(t)]^{2\beta-1}) = \sum_{n=0}^{\infty} p_n^{(\alpha-\beta n)}(x)t^n, \tag{2}$$

where $u(t)$ is the inverse of $v(t) = t(1+t)^{\beta-1}$, that is, $v(u(t)) = u(v(t)) = t$.

Proof: Let

$$(1-t)^{-\alpha}F(x,t) = \left\{\sum_{n=0}^{\infty}\binom{-\alpha}{n}(-1)^n t^n\right\}\left\{\sum_{n=0}^{\infty}p_n(x)t^n\right\}$$

$$\sum_{n=0}^{\infty}p_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\alpha}{n}(-1)^n t^{n+k}p_k(x),$$

On making the use of $\binom{-\alpha}{n} = (-1)^n\binom{\alpha+n-1}{n}$, for positive integers α and n.

$$\sum_{n=0}^{\infty}p_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+n-1}{n}p_k(x)t^{n+k}, \tag{3}$$

we get

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{\alpha+n-k-1}{n-k}p_k(x),$$

On setting α by $\alpha-\beta n$, in equation (3),yields

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\beta n)}(x)t^n = \sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty}\binom{\alpha+\beta k-1-(\beta-1)n}{n}(t)^n\right\}p_k(x)t^k.$$

On making the use of following identity [15]

$$\sum_{n=0}^{\infty}\binom{a+bn}{n}\left[\frac{z}{(1+z)^b}\right]^n = \frac{(1+z)^{1+a}}{1+(1-b)z},$$

and afterwords setting $a = \alpha+\beta k-1, b = -(\beta-1)$ and $z = u(t)$,this yields

$$\sum_{n=0}^{\infty}\binom{\alpha+\beta k-1-(\beta-1)n}{n}t^n = \frac{(1+u(t))^{\alpha+\beta k}}{1+\beta u(t)}.$$

This can be easily written in following form as

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\beta n)}(x)t^n = \frac{(1+u(t))^{\alpha}}{1+\beta u(t)}\sum_{k=0}^{\infty}p_k(x)\left[t(1+u(t))^{\beta}\right]^k,$$

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\beta n)}(x)t^n = \frac{(1+u(t))^{\alpha}}{1+\beta u(t)}\sum_{k=0}^{\infty}p_k(x)\left[u(t)(1+u(t))^{2\beta-1}\right]^k.$$

Thus

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\beta n)}(x)t^n = \frac{(1+u(t))^{\alpha}}{1+\beta u(t)}F(x,u(t)(1+u(t))^{2\beta-1}).$$

This leads the proof.

Lemma 2: The polynomial set $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ is generated by

$$\frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)}F(x,y,u(t)[1+u(t)]^{2(\gamma+\delta)-1}) = \sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n, \tag{4}$$

where $u(t)$ is the inverse of $v(t) = t(1+t)^{\gamma+\delta-1}$, that is, $v(u(t)) = u(v(t)) = t$.

Proof: Let

$$(1-t)^{-\alpha-\beta}F(x,y,t) = \left\{\sum_{n=0}^{\infty}\binom{-\alpha-\beta}{n}(-1)^n t^n\right\}\left\{\sum_{n=0}^{\infty}p_n(x,y)t^n\right\}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\alpha-\beta}{n}(-1)^n p_k(x,y)t^{n+k}.$$

Since

$$\binom{-\alpha-\beta}{n} = (-1)^n\binom{\alpha+\beta+n-1}{n},$$

where α, β and n are positive integers.
We get

$$\sum_{n=0}^{\infty}p_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha+\beta+n-1}{n}p_k(x,y)t^{n+k},$$
$$\sum_{n=0}^{\infty}p_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{\alpha+\beta+n-k-1}{n-k}p_k(x,y)t^n.$$

On comparing the coefficient of t^n, gives

$$p_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n}\binom{\alpha+\beta+n-k-1}{n-k}p_k(x,y).$$

On replacing α by $\alpha-\gamma n$ and β by $\beta-\delta n$, we get

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{\alpha-\gamma n+\beta-\delta n+n-k-1}{n-k}p_k(x,y)t^n.$$

On further simplification,yields

$$\sum_{n=0}^{\infty}\binom{a+bn}{n}\left[\frac{z}{(1+z)^b}\right]^n = \frac{(1+z)^{1+a}}{1+(1-b)z}.$$

Now, setting $a=\alpha+\beta+(\gamma+\delta)k-1, b=-(\gamma+\delta-1)$ and $z=u(t)$, this becomes

$$\sum_{n=0}^{\infty}\binom{\alpha+\beta+(\gamma+\delta)k-1-(\gamma+\delta-1)n}{n}[u(t)(1+u(t))^{\gamma+\delta-1}]^n = \frac{(1+u(t))^{1+\alpha+\beta+(\gamma+\delta)k-1}}{1+(\gamma+\delta)u(t)}.$$

Or

$$\sum_{n=0}^{\infty}\binom{\alpha+\beta+(\gamma+\delta)k-1-(\gamma+\delta-1)n}{n}t^n = \frac{(1+u(t))^{\alpha+\beta+(\gamma+\delta)k}}{1+(\gamma+\delta)u(t)};$$

this leads to

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n = \frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)}\sum_{k=0}^{\infty}p_k(x,y)\left[t(1+u(t))^{\gamma+\delta}\right]^k.$$

Finally we arrive at conclusion that

$$\sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n = \frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)}F(x,y,u(t)(1+u(t))^{2(\gamma+\delta)-1}).$$

This completes the proof.
To prove the theorems, we consider $p_n(x,y)$ is generated by

$$A(t)\phi\left(xH(t),yG(t)\right)=\sum_{n=0}^{\infty}p_n(x,y)t^n, \tag{5}$$

where

$$G(t)=\sum_{n=0}^{\infty}g_nt^{n+1},\qquad g_0\neq 0,$$

$$H(t)=\sum_{n=0}^{\infty}h_nt^{n+1},\qquad h_0\neq 0,$$

$$A(t)=\sum_{n=0}^{\infty}a_nt^{n},\qquad a_0\neq 0.$$

On taking $F(x,y,t)=A(t)\phi\left(xH(t),yG(t)\right)$, we get

$$\frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)}A(u(t)(1+u(t))^{2(\gamma+\delta)-1})\phi\left(xH(u(t)(1+u(t))^{2(\gamma+\delta)-1}),yG(u(t)(1+u(t))^{2(\gamma+\delta)-1})\right)$$

$$=\sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n.$$

Hence, we can say that if $p_n(x,y)$ is a generalized Appell set then $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ is also generalized Appell set.

Theorem 1: if $p_n(x,y)$ is Sheffer A-type zero polynomials in two variables then $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ is also Sheffer A-type zero polynomials in two variables.

Proof: Let $p_n(x,y)$ be of Sheffer A-type zero polynomials in two variables and there exists a differential operator $J=J(D)=\sum_{k=0}^{\infty}c_kD^{k+1},\ c_0\neq 0,\ D=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$, where c_k are constants, such that $Jp_n(x,y)=p_{n-1}(x,y)$,for all $n\geq 1$.
Since $p_n(x,y)$ is of A-type zero iff $p_n(x,y)$ have the generating relation [2] as

$$A(t)\exp\left(xH(t)\right)\exp\left(yG(t)\right)=\sum_{n=0}^{\infty}p_n(x,y)t^n. \tag{6}$$

From lemma 2 and equation (6), we get

$$\frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)}A(u(t))\exp\left(xH(u(t)(1+u(t))^{2(\gamma+\delta)-1}\right)\exp\left(yG(u(t)(1+u(t))^{2(\gamma+\delta)-1})\right)$$

$$=\sum_{n=0}^{\infty}p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)t^n.$$

Theorem 2: If $p_n(x,y)$ is a generalized Sheffer set of A-type zero then $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ is also generalized Sheffer set of A-type zero.

Proof: Since $p_n(x,y)$ is a generalized Sheffer set of A-type zero and the generating function is given by [18]

$$\sum_{i=1}^{r}A_i(t)\exp\left((xH(\varepsilon_i(t))\exp\left((yG(\varepsilon_i(t))\right.\right.=\sum_{n=0}^{\infty}p_n(x,y)t^n, \tag{7}$$

where

$$G(t) = \sum_{i=1}^{\infty} g_i t^i, \qquad g_1 \neq 0,$$

$$H(t) = \sum_{i=1}^{\infty} h_i t^i, \qquad h_1 \neq 0,$$

$$A_s(t) = \sum_{i=0}^{\infty} \alpha_i^{(s)} t^i, \qquad \text{(not all } \alpha_0^{(s)} \text{ are zeros)}$$

On applying lemma 2 , equation (7) takes following form

$$\sum_{n=0}^{\infty} p_n^{(\alpha-\gamma n,\beta-\delta nn)}(x,y)t^n = \frac{(1+u(t))^{\alpha+\beta}}{1+(\gamma+\delta)u(t)} \sum_{i=1}^{r} \Big[A_i(u(t)(1+u(t))^{2(\gamma+\delta)-1})$$

$$\exp\left(xH(\varepsilon_i u(t)(1+u(t))^{2(\gamma+\delta)-1}\right) \exp\left(yG(\varepsilon_i u(t)(1+u(t))^{2(\gamma+\delta)-1})\right)\Big]$$

Thus, we can say that $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ is also generalized Sheffer set of A-type zero.

The operator $J = \sum_{k=0}^{\infty} c_k D^{k+1}$ is associated with $p_n(x,y)$, where $D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. This is generated by the function $J(t) = \sum_{k=0}^{\infty} c_k t^{k+1}$ and $J(t)$ is the inverse of the function $(H+G)(t)$. The $p_n^{(\alpha-\gamma n,\beta-\delta n)}(x,y)$ corresponds to the operator which is generated by the inverse of function $(H+G)(u(t))$.

References

[1] W. A. Al-Salam, A. Verma, Generalized Sheffer Polynomials, Duke Math. J. 37 (1970) 361-365.
[2] B. Alidad, On some problems of Special Functions and Structural Matrix Analysis, Ph.D. thesis, Aligarh Muslim University, Aligarh, India, 2008.
[3] R. P. Boas Jr., R. C. Buck, Polynomial Expansions of Analytic Functions, Rev. ed., Academic Press, New York, 1964.
[4] H. Bor, H. M. Srivastava, W. T. Sulaiman, A new application of certain generalized power increasing sequences, Filomat 26 (4) (2012) 871-879.
[5] G. Bretti, C. Cesarano, P. E. Ricci, Laguerre-type exponentials and generalized Appell polynomials, Computers Math. Appl. 48 (2004) 833-839.
[6] J. W. Brown, On the Sheffer A-type of certain Modified Polynomial sets, Proc. Amer. Math. Soc. 23 (1969) 718-722.
[7] G. Dattoli, M. Migliorati, H. M. Srivastava, Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials, Math. Comput. Modelling 45 (2007) 1033-1041.
[8] G. Dattoli, Appell Polynomial Series Expansions, International Mathematical Forum 5 (14) (2010) 649-662.
[9] G. B. Djordjević, Polynomials of the Laguerre type, Filomat 26 (2) (2012) 261-267.
[10] J. L. Goldberg, On the Sheffer A-type of polynomials generated by $A(t)\psi(xB(t))$, Proc. Amer. Math. Soc. 17 (1966) 170-173.
[11] W. N. Huff, E. D. Rainville, On the Sheffer A-type of polynomials generated by $\phi(t)f(xt)$, Proc. Amer. Math. Soc. 3 (1952) 296-299.
[12] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomial in one variable, Cambridge University Press, UK, 1984.
[13] E. B. McBride, Obtaining Generating Functions, Springer, New York, 1971.
[14] V. B. Osegove, Some External Properties of Generalized Appell Polynomials, Soviet Math. 5 (1964) 1651-1653.
[15] G. Pólya, G. Szegö, Theorems and Problems in Analysis I, Springer-Verlang, Berlin, 1964.
[16] E. D. Rainville, Special Functions, the Macmillan Company, New York, 1960.
[17] I. M. Sheffer, Note on Appell polynomials, Bull. Amer. Math. Soc. 51 (1945) 739-744.
[18] A. K. Shukla, S. J. Rapeli, An Extension of Sheffer Polynomials, Proyecciones Journal of Mathematics 30 (2) (2011) 265-275.
[19] H. M. Srivastava, H. L. Manocha, A Treatise on Generating Functions, Halsted Press, New York, 1984.
[20] Subuhi Khan, Nusrat Raza, Families of Legendre-Sheffer polynomials, Math. Comput. Modelling 55 (2012) 969-982.
[21] C. J. Thorne, A property of Appell Sets, Amer. Math. Monthly 52 (1945) 191-193.
[22] H. Youn, Y. Yang, Differential Equation and Recursive Formulas of Sheffer Polynomial Sequences, ISRN Discrete Mathematics 2011 (2011) 1-16.